Topology of Networks in Generalized Musical Spaces

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ABSTRACT

The abstraction of musical structures as mathematical objects in a geometrical space is one of the major accomplishments of contemporary music theory. The author generalizes the concept of musical spaces as networks and derives compositional design principles via network topology analysis. This approach provides a framework for analysis and quantification of similarity of musical objects and structures and suggests a way to relate such measures to human perception of different musical entities. Finally, network analysis provides alternative ways of interpreting the compositional process by quantifying emergent behaviors with well-established statistical mechanics. Interpreting the latter as probabilistic randomness in the network, the author develops novel compositional design frameworks.

Network analysis methods exploit the use of graphs or networks as convenient tools for modeling relations in large data sets. If the elements of a data set are thought of as nodes, then the emergence of pairwise relations between them—edges—yields a network representation of the underlying set. Like social networks, biological networks and other well-known real-world complex networks, entire data sets of musical structures can be treated as a network, where a node represents each individual musical entity (pitch class set, chord, rhythmic progression, etc.), and a pair of nodes is connected by a link if the two objects exhibit a certain level of similarity according to a specified quantitative metric. Pairwise similarity relations between nodes are thus defined through the introduction of a measure of "distance" in the network: a "metric" [1]. As in more well-known social or biological networks, individual nodes are connected if they share a certain property or characteristic (e.g. in a social network people are connected according to their acquaintances, collaborations, common interests, etc.) Clearly, different properties of interest can determine whether a pair of nodes is connected; therefore, different networks connecting the same set of nodes can be generated.

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Network representations of musical structures are not new: From the circle of fifths [2] to the Tonnetz [3] and recent works on the spiral array model of pitch space [4], the geometry of musical chords [5] and generalized voice-leading spaces [6,7], music theorists, musicians and composers have been investigating how these structures can be combined in explaining the relations between harmony and counterpoint, the foundations of Western music. The original contribution of this article is the introduction of the representation of musical spaces as large-scale statistical mechanics networks: Uncovering their topological structure is a fundamental step to understanding their underlying organizing principles and unveiling how classifications or rule-based frameworks (for instance, common-practice harmony) can be interpreted as emerging phenomena in a complex network system.

Below, I first illustrate my approach by introducing two different metrics in the pitch class space: one based on the concept of distance between "interval vectors," the other on the distance in voice-leading space. I then discuss the definition of "rules" in the musical space and introduce the idea of composition as emerging behavior in a complex network. A discussion on the extension of this method to the classification and analysis of rhythmic progressions is presented in the online supplemental materials [8]. An ongoing study on timbre, which requires a representation not only of perceived fundamental frequencies but also of all component partials, will be the subject of a future publication.

NETWORKS IN GENERALIZED MUSICAL SPACES: PITCH CLASS SETS

Most existing approaches to the geometry of musical spaces focus on harmonic relations among pitches and attempt to define the interrelations within musical progressions as a collection of transformations obtained by the application of the five operations of octave equivalence (O), permutation (P), transposition (T), inversion (I) and cardinality change (C) to a given pitch class set [9].

To build a musical network for the pitch space, I start from the *ansatz* that the totality of a musical space can be con-

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structed by an all-combinatorial approach: the super-set of all possible tuples of NC (NC = cardinality) integers out of the total number of pitches, NP, with NC = 1, . . . ,NP—mathematically enumerated by the binomial coefficient. Obviously, these are very large spaces: The traditional chromatic set of the 12-tone equal temperament system (12TET) produces 4,096 possible choices; extending to quarter tones (24TET) we have 16,777,216 combinations, and from the 88 keys of a piano we can produce a staggering 3.1×10^{26} combinations, three orders of magnitude more than the number of units in one mole of any substance (Avogadro's number). These combinations do not allow repetitions of pitches; thus, they initially form a drastic geometrical abstraction of the totality of phase-space available for music creation. Further abstractions based on a variety of considerations can reduce the dimension of these spaces. In music theory one relies on the five OPTIC transformations to define classes of independent pitch sets [10,11] and derive classifications that can be used as analytical or compositional tools. By imposing such equivalence classes, one can reduce the 12TET combinatorial space to a mere 238 pitch class sets [12], or the 24TET to 365,588 sets. This approach can equally describe any arbitrary note sets or tuning or temperament systems.

Any set of operations that abstracts the super-set of the musical space defines a "dictionary" of the set space. This is an ordered list of {label, pitch set, . . . other descriptors of the set} elements that is exhaustive of all allowed combinatorial possibilities for that space. The elements of the dictionary can then be interpreted as the "nodes" of a deterministic (synthetic) network by defining a proper metric within that space. Here, "label" represents the generalization of the Forte classification scheme [13] for arbitrary NP, by sorting the combinatorial sets in ascending order, and eventually identifying distinct sets that share one or more identical descriptors, as in Z-related sets (pitch class set [pcs] with the same interval vector but a different prime form).

To navigate this network, I first introduce a metric based on the Euclidean distance (a generalized multidimensional Pythagorean theorem) between vectors of integers: Here I choose the interval vectors as nodes of the network, the array of natural numbers that summarize the intervals present in a pitch class set, one of its fundamental descriptors (other descriptors can be defined along similar lines [14] but will not be considered here). See the online supplements for a formal definition of this distance operator.

Given the integer character of the underlying vectors, distances are "quantized": Only discrete values are allowed. This observation suggests the introduction of a new class of operators, $\mathbf{O}(\{n_i\})$ (see online supplements), that raise or lower by an integer n the ith component of such vector. These operators play a major role in the network analysis that follows.

As an illustration of this technique, Fig. 1 shows the network of all the seven note scales that can be constructed in the 24TET super-set: 7,478 unique nodes (prime forms—

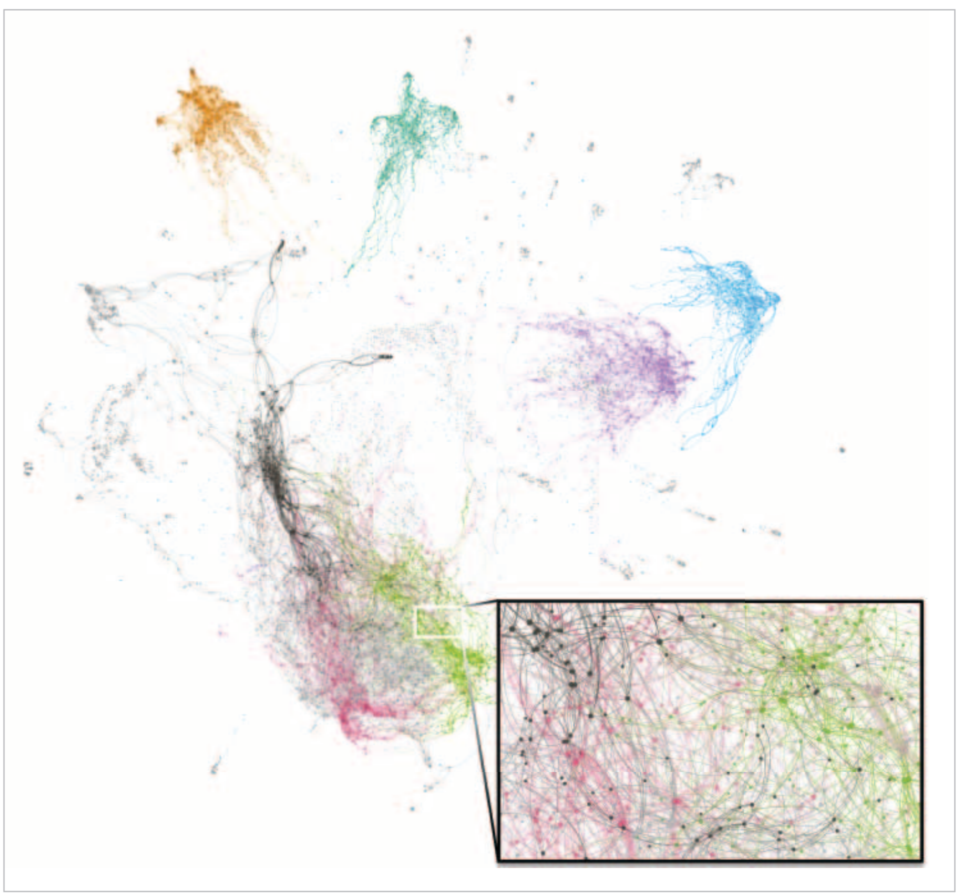

Fig. 1. Interval vector-based network of all seven note scales that can be constructed in the 24TET super-set for distances *d* (nearest neighbors only). Nodes are grouped by the degree of connections (number of connections per node) and coded according to their modularity class (see text). The inset shows the high degree of interconnection even for a single distance threshold. (© Marco Buongiorno Nardelli)

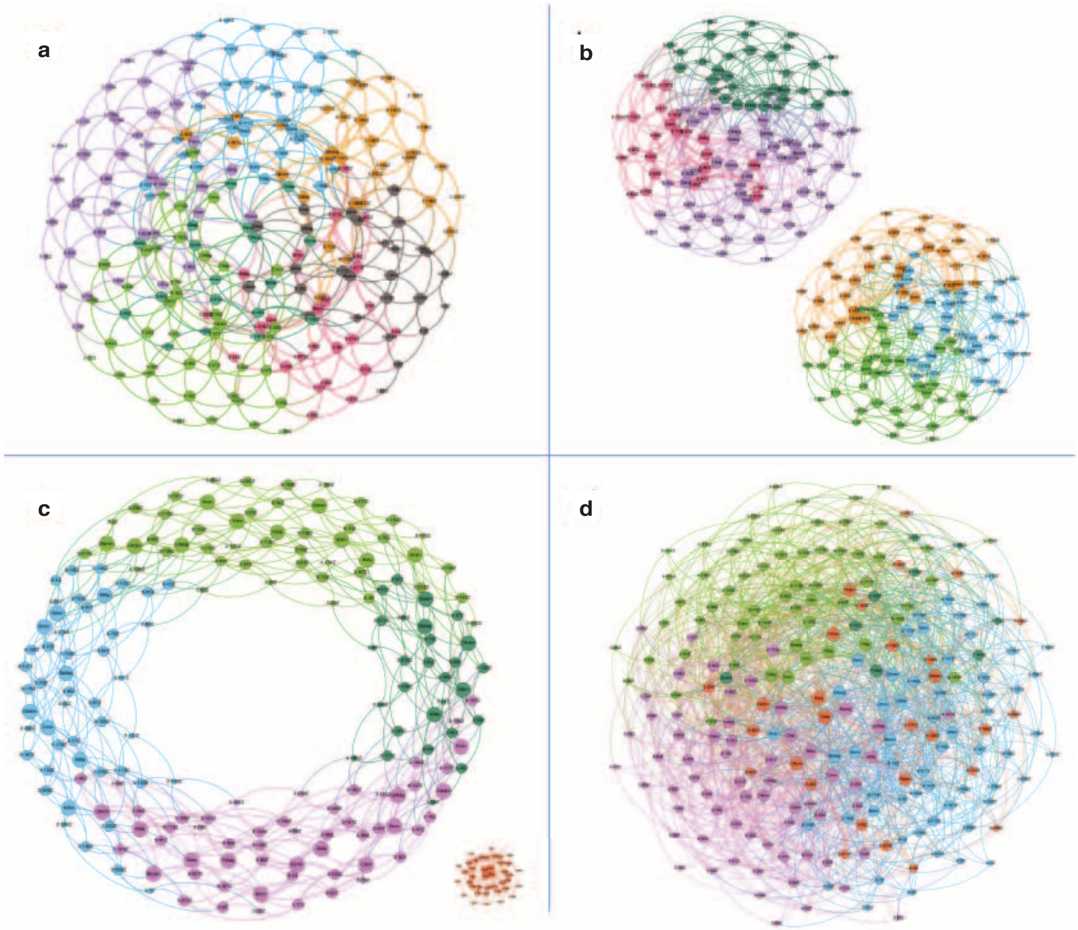

Fig. 2. Networks of the panchromatic three-part voice-leading space in 12TET. (a) $d(x,y)=1-\mathbf{O}(1)$; (b) $d(x,y)=\sqrt{2}-\mathbf{O}(1,1)$; (c) $d(x,y)=\sqrt{3}-\mathbf{O}(1,1,1)$; (d) $d(x,y)=\sqrt{5}-\mathbf{O}(1,2)$. (© Marco Buongiorno Nardelli)

OPTI equivalence classes [15,16]) with 1,715 Z-related sets. I map this network for distances $d=\sqrt{2}$, or equivalently, by applying the $\mathbf{O}(1,1)$ operator to the interval vector, which raises or lowers two components by 1.

Although the above classification could have been obtained also by other mathematical means, the novelty here is that having a network representation of a musical space allows us to apply well-established techniques of statistical mechanics for the analysis of large-scale networks and to quantitatively examine the structure of relationships between pitch classes. Indeed, given a network we can perform many statistical operations that shed light on the internal structure of the data. In this work I consider only two of such measures, degree centrality and modularity class. The degree of a node is measured by the number of edges that depart from it. It is a local measure of the relative "importance" of a node in the network. Modularity is a measure of the strength of division of a network into communities: High modularity (above 0.6 in a scale from 0 to 1) corresponds to networks that have a clearly visible community structure [17]. In the case of the network in Fig. 1, I measure an average degree of 5.33 and a modularity of 0.865, clearly manifest in the high degree of separation between regions of different colors. Isolating communities provides a way to operate within regions of higher similarity, and thus in this particular case, of closer harmonic content.

NETWORKS IN GENERALIZED MUSICAL SPACES: VOICE LEADING

Let's now move to the construction of networks based on the voice-leading distance measure (see the online supplements for the definition of this metric). Here the nodes of the network represent individual pcs (chords), and their relationships (edges) are defined by their distance in the harmonic space. To illustrate, I start by restricting analysis to the space of all triads in 12TET. First consider the super-set of octave-equivalent normal forms (prime forms plus inversions—OPT equivalence classes). In this space, a major and a minor chord are considered different although they can be reduced to the same prime form. This is the space that best abstracts three-part counterpoint in common-practice harmony, where the $\mathbf{O}(\{n_i\})$ operators act directly on the pcs as voice leading operators. The network restricted to nearest neighbors ($\mathbf{O}(1)$) is isomorphic to the orbifold of the quotient space $\mathbb{T}2/\mathcal{S}3$ as derived in Callender [18] and shown in the online supplement. This result demonstrates clearly the power of our approach: Our network analysis is confirming relations between sets when these are known, a proof of its reliability and, at the same time, provides a direct method to explore musical spaces beyond the simplest abstractions.

To illustrate this argument further, let's release the constraint on the transposition equivalence class and derive the network of the panchromatic three-parts voice-leading space in 12TET. The four panels in Fig. 2 show networks sliced at different distance thresholds (data to reproduce every network are available as the online supplements). Network (a) is the orbifold of minimal distance neighbors. In a common practice harmony framework, it is the extension of the $\mathbb{T}2/\mathcal{S}3$ orbifold to every major and minor key. Network (a) displays an average node degree of 4.9 with a modularity index of 0.56. Network (b) is the network with edges at d corresponding to the operator $\mathbf{O}(1,1)$. Interestingly, for operators $\mathbf{O}(\{n_i\})$ of higher order, the topology of the network can be drastically altered. In this case, the network is split into two disconnected orbifolds that have as centers the augmented C,D and C♯,E♭ triads, respectively, with an average degree of 8.8 and a modularity of 0.57.

Similarly, in (c) the network, defined by d, is split into a large torus that excludes all augmented triads and their close relatives (degree = 5.9, modularity = 0.63); finally, in (d) the topology of the orbifold is recovered by $d(x,y)=2-\mathbf{O}(2)$, with degree = 13.94 and modularity = 0.26. The structure of the network built under any of the $\mathbf{O}(\{n_i\})$ operators reflects particular classes of functional properties of chord progressions. Table 1 summarizes some of the structures outlined in networks a–d. One must note that the $\mathbf{O}(\{n_i\})$ operators contain and generalize the operators of the neo-Riemannian triadic theory: P,L=$\mathbf{O}(1)$, R=$\mathbf{O}(2)$, N,S=$\mathbf{O}(1,1)$, H=$\mathbf{O}(1,1,1)$ [19]. More generally, any progression in a given musical space can be obtained by the successive application of $\mathbf{O}(\{n_i\})$ operators, thus creating the desired sequence.

Finally, the distance operators find a direct application in the definition of "parsimony," as the average of the weight (inverse of the distance between two chords) along the progression provides a measure of the "motion" of the sequence. See the online supplemental material for a more detailed discussion.

TABLE 1. Example of the functional chord progressions from selected distance operators.

Operator	Distance	Example of functional chord progression
O(1)	1	aug ⇒ Maj ⇒ min ⇒ dim
O(2)	2	Maj to 7 progression (i.e. C ⇒ C7)
O(1,2)	$\sqrt{5}$	IV-I progression
O(1,1,2)	$\sqrt{3}$	V7-I progression
O(1,2,2)	3	I-ii, viio-I and IV-V progression

"COMPOSITION" AS EMERGENT BEHAVIOR IN A COMPLEX NETWORK

All the networks in Figs 1 and 2 are "synthetic": They are the result of the deterministic application of equivalence classes or other discrete operators to an all-combinatorial super-set. As such they provide a complete account of the structure and topology of arbitrary musical spaces but add no additional understanding beyond their geometrical structure. In the process of music-making, the composer necessarily builds their own harmonic and melodic sequence by making specific choices on the underlying network structure based on a wide variety of considerations: from aesthetic to functional, personal or programmatic. Moreover, the network is generally already sliced according to generic rules of harmonic progression, where, for instance, only selected edges are allowed (i.e. common practice harmony).

Thus the deterministic fabric of the absolute geometrical space is shredded by probabilistic choices: The network becomes complex. This is the process at the core of the creation of any musical work: The composer, by introducing probabilistic choices in the creation of their own version of the network, induces emergent behaviors that manifest themselves in the harmonic and contrapuntal framework of the piece.

One can directly observe all these concepts in the analysis of a score: Figure 3 shows the network representations of two works of contrasting characteristics: J.S. Bach's chorale from the cantata *Erfreut euch, ihr Herzen*, BWV 66, and the sixth movement (*sehr langsam*) of A. Schoenberg's *Sechs Kleine Klavierstücke*, Op. 19. The edges here are directional, to indicate the pcs progression in Bach's piece. A simple statistical analysis shows that Bach's chorale network, Fig. 3a, has an average degree of 1.70 per node, and a modularity index of 0.46: The distribution of edges is very sparse, and many nodes are visited numerous times. The modularity index clearly individuates the broad tonal areas visited in the short piece as different shades in Fig. 3a (in color in digital publication): We start from A major, modulate to F♯ minor, then back to A major with a pass-through C♯ minor, then C♯ major and finally ending in F♯ major (see video, online supplements). Analysis of the distribution of the $\mathbf{O}(\{n_i\})$ operators (Fig. 3b) highlights the predominance of $\mathbf{O}(1,2)$ and $\mathbf{O}(2)$, $\mathbf{O}(1)$, $\mathbf{O}(1,1,2)$ and $\mathbf{O}(1,2,2)$ in the voice leading space (see Table 1). These conclusions are further supported by analysis of the full corpus of the 371 Bach chorales (online supplements).

In contrast, Schoenberg's network displays a strikingly different topology (Fig. 3c): The edges form a loop and each node has many fewer connections (average degree 1.27). The progressions are largely chromatic, with $\mathbf{O}(1)$ and $\mathbf{O}(1,1,1)$ used extensively (Fig. 3d) and coexist with leaps of large distances in the voice leadings, corresponding to movements mostly from single pitches to large chords—an evidence of the broad range of cardinalities in the pcs (between 1 and 8

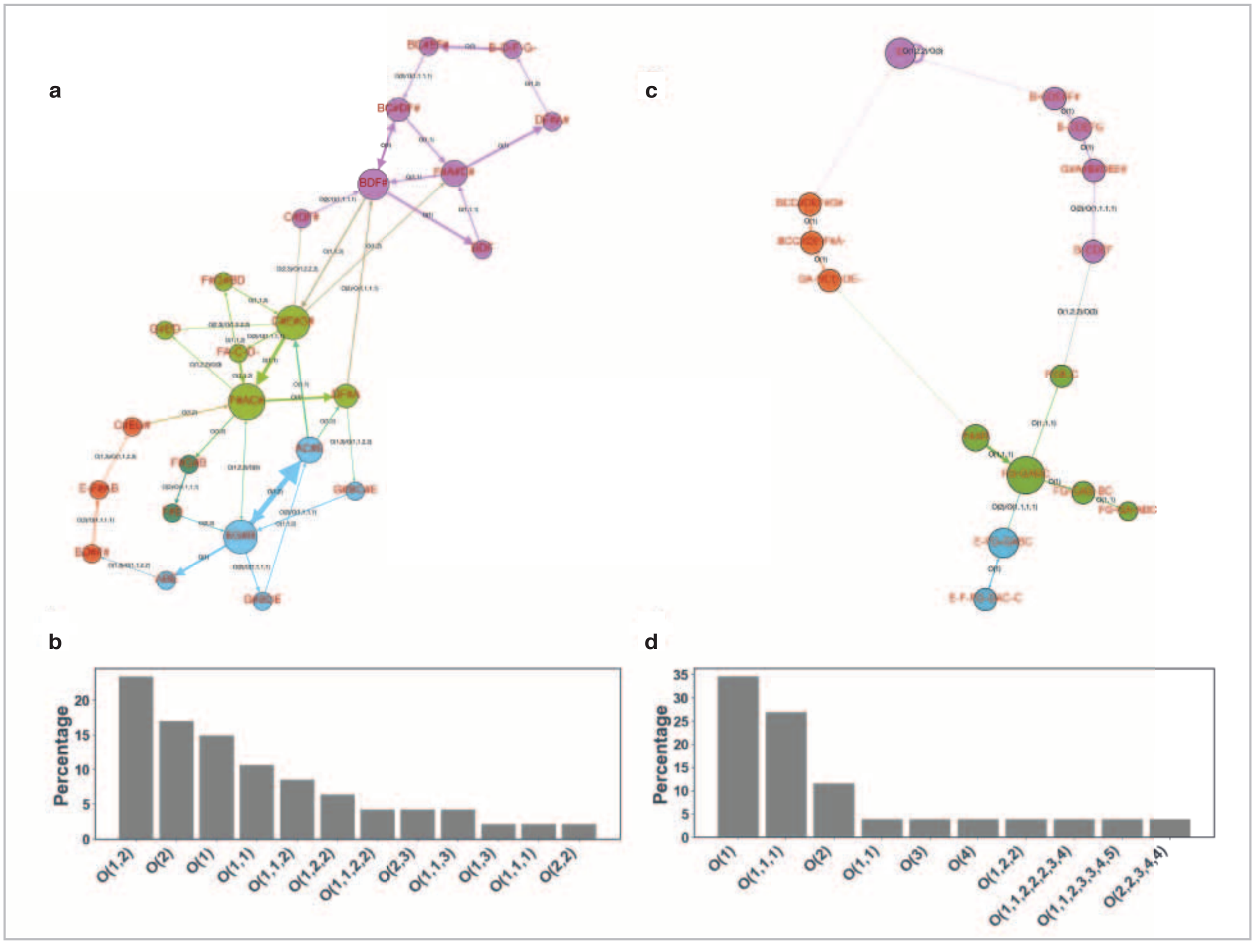

Fig. 3. (a) Voice-leading network of J.S. Bach's chorale from the cantata *Erfreut euch, ihr Herzen*, BWV 66 (1724); (b) probability distribution of occurrences of specific distance operators in the chorale; (c) voice-leading network of the sixth movement (*sehr langsam*) of A. Schoenberg, *Sechs Kleine Klavierstücke*, Op. 19 (1913); (d) probability distribution of occurrences of specific distance operators in Schoenberg's piece. A video animation of the evolution of both progressions is available in the online supplements. (© Marco Buongiorno Nardelli)

versus the largely triadic harmony of Bach's chorale). Clearly, new relations that replace the framework of classical harmony emerge from this network. However, a modularity of 0.48 demonstrates the resilience of a compositional design that is very classical: The work is still centered strongly on a harmonic center (the incomplete dominant seventh chord F#AB—Forte class 3-7) and uses the FF#GABC cluster (Forte class 6-Z12) as a pivot point of most progressions (see video, online supplements).

CONCLUSIONS

The analytical process outlined above is first and foremost a methodology developed for my own compositional practice. It can be clearly generalized for the generation of algorithmically based music composition agents: Current generators for complex networks that include, among others, the well-known models of Erdős-Rényi (probability of edge creation) [20] or Barabási-Albert (probabilistic distribution of the number of edges to attach from a new node) [21] can be used for the generation of such complex networks. For instance, using the Barabási-Albert algorithm of preferential attachment with a probability distribution of my choice, I can generate a sliced network that would correspond to totally new harmonic progression rules and, from there, generate innumerable variations on similar structures. This approach provides internal coherence within the framework and, eventually, could lead to novel pathways for the design of algorithmic agents of automatic composition [22]. These procedures are becoming essential in my artistic research. As an illustration, the reader can find the score and the compositional notebook of *Le Reseau de Ton Souvenir*, a suite for solo alto recorder that uses, at its foundation, harmonic hierarchies and rhythmic sequences based on the complex network analysis outlined above.

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Data and Materials Availability

All data is available in the main text or as supplementary material. The Python modules for the generation and analysis of networks in generalized musical spaces, musicntwrk, are freely available at www.musicntwrk.com.

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